

Propagation of Bessel beams in absorbing media: a new generation of GPR devices?

D. Mugnai and P. Spalla

*Nello Carrara" Institute of Applied Physics, CNR Florence Research Area,
Via Madonna del Piano 10, 50019 Sesto Fiorentino (FI), Italy*

In recent years localized waves have aroused great interest in the scientific community [1]. In relation to this topic, many efforts were devoted to the analysis of Bessel beams because of their unusual features: they are non-diffracting [2, 3, 4, 5, 6] and show superluminal behavior both in phase and group velocities [7, 8, 9, 10, 11].

The aim of this paper is the study of the propagation of a Bessel beam through two absorbing layers, limited by two different half-spaces. Our approach will be based on the scalar analysis, since this analysis was proved to be an excellent approximation of the vectorial field which describes a Bessel beam [12].

Let us consider a system formed by two layers, 1 and 2, limited by two half-spaces denoted by 0 and 3, as sketched in Fig. 1.

All planes limiting the four different media are parallel and of infinite length. A localized wave, at frequency ω , impinges on medium 1 coming from medium 0, which is non absorbing. On the contrary, media 1, 2, and 3 are absorbing. Be d_1 and d_2 the thickness of the layer 1 (medium 1) and 2 (medium 2) respectively. Let us denote with the subscripts i , r and t the incident, reflected and transmitted fields, respectively. The subscripts 0, 1, 2, and 3 characterized the media in which the field propagates. Be n_0 , n_1 , n_2 , n_3 the refractive indices of media 0, 1, 2, 3, respectively. For the time being, let us consider an ideal perfect homogeneous system.

Let us consider a localized wave of Bessel type which impinges in medium 1 at normal incidence. We wish to remind that a Bessel wave - or Bessel beam - originates by the interference of an infinite number of plane waves, each of which tilted of the same angle θ_0 with respect to a given direction, say z , which is the direction of propagation of the beam. We refer to a specific Bessel beam, namely the one described in Ref. [12]. For a Bessel beam like the one described in this particular reference, the field is linearly polarized and the scalar approximation is justified since two components of the field are negligible as compared to the third component, which is the dominant one. With reference to a cylindrical coordinate system (ρ, ψ, z) , a Bessel beam E is given by

$$E = J_0(nk_0\rho \sin \theta_0) e^{ink_0 z \cos \theta_0}, \quad (1)$$

where J_0 is the Bessel function of first type, θ_0 is the axicone angle which characterizes the Bessel beam, n is the refractive index of the medium, and $k_0 = \omega/c$ is the wave number in vacuum (the field E is rotationally symmetric and thus independent on the angular coordinate ψ). Thus, the electric fields in media 0, 1, 2, 3 can be written as (see Fig. 1)

$$E_{0i} = J_0 e^{ik_0 n_0 z \cos \theta_0}, \quad (z \leq 0) \quad (2)$$

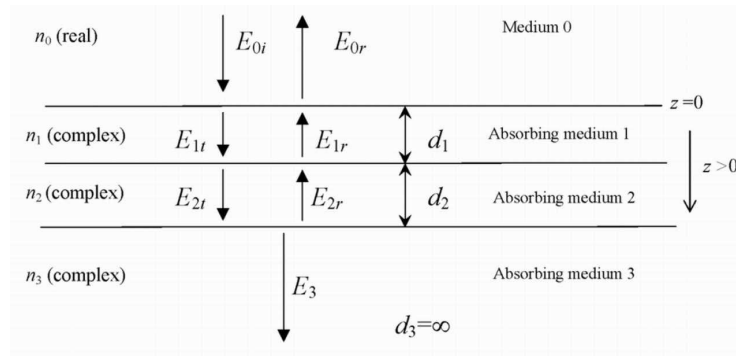


FIG. 1: Schematic representation of propagation. A Bessel beam impinges on medium 1 at normal incidence, then in medium 2 and, lastly, propagates through medium 3. Media 1,2 and 3 are absorbing media.

$$E_{0r} = r_0 J_0 e^{-ik_0 n_0 z \cos \theta_0}, \quad (z \leq 0) \quad (3)$$

$$E_{1t} = t_1 J_{01} e^{ik_0 n_1 z \cos \theta_1}, \quad (0 \leq z \leq d_1) \quad (4)$$

$$E_{1r} = r_1 J_{01} e^{-ik_0 n_1 z \cos \theta_1}, \quad (0 \leq z \leq d_1) \quad (5)$$

$$E_{2t} = t_2 J_{02} e^{ik_0 n_2 (z-d_1) \cos \theta_2}, \quad (d_1 \leq z \leq (d_1 + d_2)) \quad (6)$$

$$E_{2r} = r_2 J_{02} e^{-ik_0 n_2 (z-d_1) \cos \theta_2}, \quad (d_1 \leq z \leq (d_1 + d_2)) \quad (7)$$

$$E_3 = t_3 J_{03} e^{ik_0 n_3 (z-(d_1+d_2)) \cos \theta_3}, \quad (z \geq (d_1 + d_2)) \quad (8)$$

where

$$J_0 = J_0(k_0 n_0 \rho \sin \theta_0), \quad J_{01} = J_0(k_0 n_1 \rho \sin \theta_1), \quad J_{02} = J_0(k_0 n_2 \rho \sin \theta_2) \quad J_{03} = J_0(k_0 n_3 \rho \sin \theta_3),$$

and θ_1 , θ_2 , and θ_3 are the complex angles which characterize the Bessel function inside the media 1, 2 and 3, respectively. The temporal factor $e^{-i\omega t}$, which is present in all fields, is omitted for the sake of simplicity.

All the refractive indices are complex, with the exception of n_0 . The quantities r , t , which are labeled with reference to the medium in which the field propagates, denote the complex reflection and transmission coefficients, respectively.

It is crucial to note that, even if the refractive index n_1 and the angle θ_1 are both complex, the product $n_1 \sin \theta_1$ is real [13]. In fact, in the propagation of a plane wave in a conducting medium, the surfaces of constant amplitude are parallel to the interface [14]. Thus, in order to meet this condition the product $n_1 \sin \theta_1$ has to be real, otherwise the constant-amplitude surfaces would be plane, but with an inclination with respect to the interface. By considering the continuity of the phase of the incident and reflected waves, it is also possible to demonstrate that $n_1 \sin \theta_1 = n_0 \sin \theta_0$ [13].

Since all the products $n_n \sin \theta_n$ are real and are equal to $n_0 \sin \theta_0$, we come to the remarkable conclusion that the shape of the Bessel beam is not modified when propagating into an absorbing medium. As a consequence, Eqs. (2)-(8) can be written as

$$E_{0i} = J_0 e^{ik_0 n_0 z \cos \theta_0}, \quad (9)$$

$$E_{0r} = r_0 J_0 e^{-ik_0 n_0 z \cos \theta_0}, \quad (10)$$

$$E_{1t} = t_1 J_0 e^{ik_0 n_1 z \cos \theta_1}, \quad (11)$$

$$E_{1r} = r_1 J_0 e^{-ik_0 n_1 z \cos \theta_1}, \quad (12)$$

$$E_{2t} = t_2 J_0 e^{ik_0 n_2 (z-d_1) \cos \theta_2}, \quad (13)$$

$$E_{2r} = r_2 J_0 e^{-ik_0 n_2 (z-d_1) \cos \theta_2}, \quad (14)$$

$$E_3 = t_3 J_0 e^{ik_0 n_3 (z-(d_1+d_2)) \cos \theta_3}. \quad (15)$$

In order to evaluate the reflection and transmission coefficients, we shall apply the boundary conditions to the field (that is the condition of continuity of the field on the border surface between two media), and to its first derivative with respect to the direction of propagation. The latter condition works only in the scalar approximation, and is none other than the condition of continuity of the tangential component of the magnetic field. In formula we have:

$$\begin{aligned} [E_{0i} + E_{0r}]_{z=0} &= [E_{1t} + E_{1r}]_{z=0} \\ [E_{1t} + E_{1r}]_{z=d_1} &= [E_{2t} + E_{2r}]_{z=d_1} \\ [E_{2t} + E_{2r}]_{z=(d_1+d_2)} &= [E_3]_{z=(d_1+d_2)}, \end{aligned} \quad (16)$$

and

$$\begin{aligned} \left[\frac{\partial E_{0i}}{\partial z} + \frac{\partial E_{0r}}{\partial z} \right]_{z=0} &= \left[\frac{\partial E_{1t}}{\partial z} + \frac{\partial E_{1r}}{\partial z} \right]_{z=0} \\ \left[\frac{\partial E_{1t}}{\partial z} + \frac{\partial E_{1r}}{\partial z} \right]_{z=d_1} &= \left[\frac{\partial E_{2t}}{\partial z} + \frac{\partial E_{2r}}{\partial z} \right]_{z=d_1} \\ \left[\frac{\partial E_{2t}}{\partial z} + \frac{\partial E_{2r}}{\partial z} \right]_{z=(d_1+d_2)} &= \left[\frac{\partial E_3}{\partial z} \right]_{z=(d_1+d_2)}. \end{aligned} \quad (17)$$

With the notations

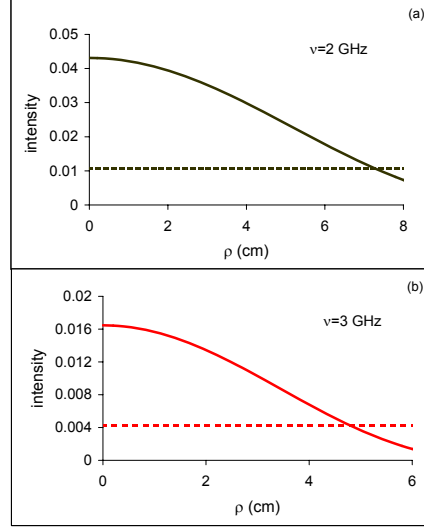


FIG. 2: Intensity of the reflected field in medium 0 (Eq. 3), as a function of the radial coordinate ρ . Near $\rho \simeq 0$ the intensity of the Bessel beam is greater than that of the plane wave. On going away from the origin, the intensity of the beam decreases and the intensity of plane wave surmounts. Parameter values are: $\theta_0 = 30^\circ$, $n_{1r} = 1.1$, $n_{2r} = 2.2$, $d_1 = 25$ cm, $d_2 = 1$ cm, $\nu = 2$ GHz (a), and $\nu = 3$ GHz (b). The values of the refractive indices, n_{1r} and n_{2r} , refer to dry sandy ground (medium 1) and glass (medium 2), respectively. The $n_{2r} = 2.2$ value may also refer to other kinds of materials, such as stone or ceramics materials

$$\begin{aligned} \alpha_1 &= k_0 n_1 d_1 \cos \theta_1, & \alpha_2 &= k_0 n_2 d_2 \cos \theta_2, \\ \phi_0 &= k_0 n_0 \cos \theta_0, & \phi_1 &= k_0 n_1 \cos \theta_1, & \phi_2 &= k_0 n_2 \cos \theta_2, & \phi_3 &= k_0 n_3 \cos \theta_3, \end{aligned} \quad (18)$$

Eqs. (16) and (17) can be written as:

$$\begin{aligned} 1 + r_0 &= t_1 + r_1 \\ t_1 e^{i\alpha_1} + r_1 e^{-i\alpha_1} &= t_2 + r_2 \\ t_2 e^{i\alpha_2} + r_2 e^{-i\alpha_2} &= t_3, \end{aligned} \quad (19)$$

$$\begin{aligned} \phi_0(1 - r_0) &= \phi_1(t_1 - r_1) \\ \phi_1(t_1 e^{i\alpha_1} - r_1 e^{-i\alpha_1}) &= \phi_2(t_2 - r_2) \\ \phi_2(t_2 e^{i\alpha_2} - r_2 e^{-i\alpha_2}) &= t_3 \phi_3. \end{aligned} \quad (20)$$

By solving equations (19) and (20), we obtain the reflection coefficient r_0 related to the propagation in the incoming medium 0, and the reflection and transmission coefficients related to the propagation in the layers 1,2 and in the half-space 3 (the calculations are rather laborious but present no difficulty)

$$r_0 = \frac{a \phi_0 - a_1 \phi_1}{a \phi_0 + a_1 \phi_1}, \quad (21)$$

$$r_1 = \frac{1 + r_0 - r_2 A^- e^{-i\alpha_1}}{1 - e^{-2i\alpha_1}} \quad (22)$$

$$r_2 = - \frac{2\phi_1 e^{-i\alpha_1} (1 + r_0)}{\phi_2 A^+ (e^{-2i\alpha_1} - 1) - \phi_1 A^- (e^{-2i\alpha_1} + 1)} \quad (23)$$

$$t_1 = e^{-i\alpha_1} (r_2 A^- - r_1 e^{-i\alpha_1}) \quad (24)$$

$$t_2 = -r_2 e^{-2i\alpha_2} \Phi \quad (25)$$

$$t_3 = \frac{\phi_2}{\phi_3} (t_2 e^{i\alpha_2} - r_2 e^{-i\alpha_2}), \quad (26)$$

where

$$\begin{aligned} a &= (e^{-2i\alpha_1} - 1)^2(\phi_1 A^- + \phi_2 A^+) - 2\phi_1 A^- e^{-2i\alpha_1}(e^{-2i\alpha_1} - 1) \\ a_1 &= \left[(e^{-2i\alpha_1} - 1)(\phi_1 A^- + \phi_2 A^+) - 2\phi_1 A^- e^{-2i\alpha_1} \right] (e^{-2i\alpha_1} + 1) + 4\phi_1 A^- e^{-2i\alpha_1} \\ \Phi &= \frac{\phi_3 + \phi_2}{\phi_3 - \phi_2} \end{aligned}$$

and

$$A^+ = 1 + e^{-2i\alpha_2}\Phi, \quad A^- = 1 - e^{-2i\alpha_2}\Phi.$$

By replacing Eqs. (21)-(26) in Eqs. (9)-(15), we are now able to obtain the electromagnetic fields that describe the propagation of a Bessel beam through the multilayer system of Fig. 1.

We are mainly interested in evaluating the reflected field in medium 0, in a physical situation in which medium 3 is equal to the medium 1, that is for $n_1 = n_3$. This situation is related to the possibility of using Bessel beams in searching for objects buried underground as well as any kind of macro impurity or defect inside a given material.

Because of the interference of multiple reflections on the layers, the amplitude of the reflected field is an oscillating function with respect to the frequency. For this reason, the intensity of the beam greatly depends on the frequency. Thus, for given parameter values, the choice of the frequency is extremely important in order to have the Bessel beam amplitude surmounting the amplitude of the plane wave.

In Fig. 2 we show the intensity of the reflected field (Eq. 3) from a Bessel beam and a plane wave as a function of the radial coordinate ρ , for two different frequencies. The figure is limited to the central portion of the Bessel beam, since it is the only one of physical interest.

In order to check the validity of the scalar approximation, for the value of the parameter θ_0 used here ($\theta_0 = 30^\circ$), and for ρ in the range of physical interest, we have verified that one component of the electric field is much larger than the other. The components of the field became comparable only close to the first zero of the Bessel function, and, in this case, the scalar approximation loses its validity.

The fields which describe the propagation of the plane wave through the layers are given by Eqs. (2)-(8) by putting $\theta_0 = 0$.

Our parameter values refer to the case of a sheet of glass buried underground [15]. Near $\rho = 0$ the gain is evident for both frequencies. Even if the intensity of the reflected field is only a few percent with respect to the incoming field [16], the intensity of the Bessel beam in both cases is about three times greater than that of the plane wave.

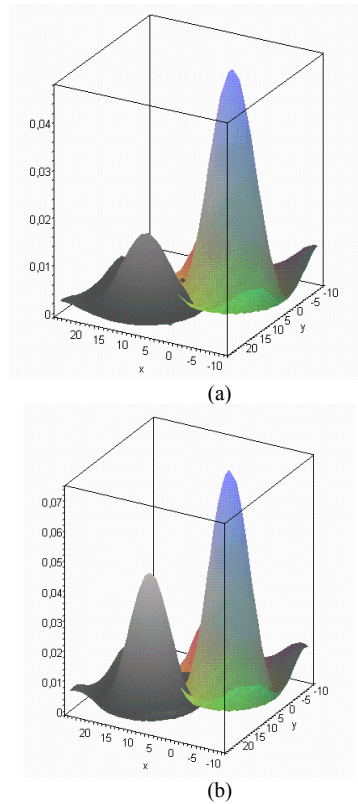
For complex refractive indices, we chose a mean standard value in the microwaves range. Moreover, we set the value of the imaginary part of the dielectric constant at approximately one order of magnitude less than the real one. We are aware that this position may appears to be rather rough. However, it is a plausible approximation for our purposes, since our aim is to analyze the difference in the propagation between a Bessel beam and a plane wave, rather than to analyze a specific material (under specific conditions of humidity, temperature etc.). We should recall that the complex refractive index $n_c = n_r + in_i$ is related to the complex dielectric constant $\epsilon_c = \epsilon' + i\epsilon''$ by means of the relation $n_c^2 = \epsilon_c$. Thus, for $\epsilon'' = 0.1\epsilon'$ we simply have $n_i \simeq 0.05 n_r$.

Once the use of Bessel beams has been shown to be advantageous with respect to that of plane waves, we can investigate the difference of intensity in the reflected field in the presence or absence of a given buried material.

In Fig. 3 we report the three-dimensional intensity of the reflected field as a function of the Cartesian coordinates x and y ($\rho = \sqrt{x^2 + y^2}$). The higher signal refers to the passage in the presence of glass, while the lower one refers to the passage only through the ground. The difference in the intensities is evident so that, in an experimental investigation, the presence of a buried object (glass, stone or ceramics materials) should be clearly detected.

We have shown that the use of Bessel beams can be advantageous as compared to plane waves, in the detection of buried objects. Our analysis refers to a particularly favorable case, that is, objects buried inside very dry material (e.g. sandy or clayey ground) [17]. In this case, in fact, microwaves undergo small absorption.

We showed that, near $\rho = 0$, Bessel beams always present a gain with respect to plane waves, provided that a suitable frequency is chosen: the gain becomes more and more negligible as the frequency increases. For high frequency values, the advantage of using a Bessel beam is lost, since the amplitude of the plane wave always exceeds that of the beam.



[h]

FIG. 3: Three-dimensional intensity of the reflected field (Eq. 3) as a function of the Cartesian coordinates x and y . Higher signal refers to the reflected field in the presence of buried object; smaller signal (dark gray) refers to the propagation through the only ground. Parameter values are: $\theta_0 = 30^\circ$, $d_1 = 25$ cm, $d_2 = 1$ cm, and (a): $n_{1r} = 1.2$, $n_{2r} = 2.25$, $\nu = 2.2$ GHz; (b): $n_{1r} = 1.4$, $n_{2r} = 2.3$, $\nu = 2.32$. We now need to make a comment or two on the size of the Bessel beam generator: for $\theta_0 = 30^\circ$, in order to have a field depth of 50 cm, the converging system must have a radius of about 30 cm [7]. This dimension is not unusual for an electromagnetic mirror, while a lens of this size would have to be custom-made. Naturally, the size of the microwave generator-launcher is different, depending on the frequency.

A second, and perhaps more important, aspect related to Bessel beams is that the difference between the propagation through the ground alone and the propagation through the ground in the presence of buried objects is quite evident (Fig. 3) [18].

In the light of these conclusions, we can think that a GPR (ground penetrating radar) system which utilizes a Bessel beam as incoming pulse could present some advantages as compared to a traditional GPR apparatus.

Our theoretical model works under two main approximations: namely, the smooth surfaces of the layers and homogeneous media. From an experimental point of view, the first approximation works well as long as the dimensions of roughness are sufficiently small as compared with the wavelength; the second one requires a specific software, in order to select between signal and noise.

Because of their feature of localized waves (i.e. localized energy), Bessel beams could also provide further information about the dimension and shape of buried objects. However, an analysis devoted to this aspect goes beyond the purpose of the present work and will be reported elsewhere.

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